13th Week Wednesday

17 Mayıs 2023 Çarşamba 14:34

MDT-II topics

- Subspaces (finding a basis for subspaces)
- Span, Linear Independence.
- Basis & dimension, Change of basis
- Linear Transformations, Kernel & Range of a L.T., Matrix Representations for L.T.
- (Eigenvalue, eigenvector.)

→ Finding Null Space, Row Space, Column Space of a matrix

1.

Let $A = \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 1 & 2 & -2 & 2 & 1 \\ 2 & 4 & -2 & 3 & 4 \end{bmatrix}$ →

(a) Find a basis for the column space of A and determine its dimension.

(b) Find a basis for the row space of A and determine its dimension.

(c) Find a basis for the null space of A and determine its dimension.

(d) What is $rank(A)$, $Null(A)$?

$N(A)$ → the solution space to $\boxed{Ax=0}$

$\rightarrow \dim(N(A)) = 3$

$$\left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & -1 & 0 \\ 1 & 2 & -2 & 2 & 1 & 0 \\ 2 & 4 & -2 & 3 & 4 & 0 \end{array}\right] \xrightarrow[-2r_1+r_3 \to r_3]{-r_1+r_2 \to r_2} \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & -1 & 0 \\ 0 & 0 & 2 & -1 & 2 & 0 \\ 0 & 0 & 6 & -3 & 6 & 0 \end{array}\right] \rightarrow \underbrace{\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 2 & -4 & 3 & -1 & 0 \\ 0 & 0 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}}_{REF(A)}$$

$x_2 = r \in \mathbb{R}$
$x_4 = s \in \mathbb{R}$ } free
$x_5 = t \in \mathbb{R}$

$x_3 = \frac{1}{2}s - t$

$x_1 + 2r - 4\left(\frac{1}{2}s - t\right) + 3s - t = 0$

$\Rightarrow x_1 = -2r - s - 3t$

$\rightarrow N(A) = $ Solution space $= \left\{ \left(-2r-s-3t, r, \frac{1}{2}s - t, s, t\right) : r, s, t \in \mathbb{R} \right\} \leq \mathbb{R}^5$

$$\begin{bmatrix} -2r-s-3t \\ r \\ \frac{1}{2}s-t \\ s \\ t \end{bmatrix} = r \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + s \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1/2 \\ 1 \\ 0 \end{bmatrix}}_{v_2} + t \underbrace{\begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{v_3}$$

$\text{Span}\{v_1, v_2, v_3\} = N(A)$

lin. independence

$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$ $\quad c_1 = c_2 = c_3 = 0 \Rightarrow$ ✓

$c_1 = 0 \quad c_2 = 0 \quad c_3 = 0$ ✓ $\Rightarrow \{v_1, v_2, v_3\}$ are linearly independent.

A basis for $N(A) \Rightarrow \{v_1, v_2, v_3\}$

$\dim(N(A)) = 3$ → Null (A)

for

A basis for

Row Space of A: All not-all-zero row vectors of REF(A).

R(A)

$$REF(A) = \begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 0 & 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A basis for Row Space of A $= \left\{ \begin{bmatrix} 1 \\ 2 \\ -4 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1/2 \\ 1 \end{bmatrix} \right\}$

$$\dim(R(A)) = 2$$

Column Space of A: Look at the column positions of leading-1's in REF(A).
Take the columns of A on those positions.

C(A)

$$REF(A) = \begin{array}{ccccc} 1^{st} & & 3^{rd} & & \\ \downarrow & & \downarrow & & \end{array}$$
$$\begin{bmatrix} 1 & 2 & -4 & 3 & -1 \\ 0 & 0 & 1 & -1/2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Take 1st and 3rd column of A.

A basis for C(A) $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ -2 \end{bmatrix} \right\}$

$$\dim(C(A)) = 2$$

Rank(A)=2

$$\text{Rank}(A) \Rightarrow \dim(R(A)) \ (= \dim(C(A)))$$

! For any matrix A, $\underbrace{\text{Rank}(A)}_{2} + \underbrace{\text{Null}(A)}_{3} = \#\text{ columns of } A = 5$

Ex/ $S = \{ (a, a+b+c+d, 2b-3c) : a,b,c,d \in \mathbb{R} \} \leq \mathbb{R}^3$

Find a basis for S.

dim(S) ≤ 3

$$\begin{bmatrix} a \\ a+b+c+d \\ 2b-3c \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + b\underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{v_2} + c\underbrace{\begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}}_{v_3} + d\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{v_4} \Rightarrow S = \text{Span}\{v_1, v_2, v_3, v_4\}$$

v_1 → can not be written as a lin. comb. of v_2, v_3, v_4

we can not throw v_1 out.

Are they lin independent? ✗

$$\frac{3}{5}v_2 + \frac{2}{5}v_3 = v_4$$

$r_1 + r_2 = 1$
$2r_1 - 3r_2 = 0$

$5r_1 = 3 \quad r_1 = 3/5 \quad r_2 = 2/5$

we can trow v_4 out.

A basis for S:

$\{v_1, v_2, v_3\}$ → a lin. indep. set which spans S.

! $\dim(V) = n$ $\quad > n$ element-set can not be linearly independent.

! $\dim(V) = n$ $> n$ element - set can not be linearly independent.

$< n$ element - set can not be a spanning set for V.

for $= n$ element - set ;

if lin. indep $\Rightarrow$ it spans. if spans $\Rightarrow$ lin. indep.

det $\neq 0$ ✓ ✓ $\Rightarrow$ it forms a basis.

(lin. indep + span)

8. Let $\mathbb{P}_3$ be the vector space of polynomials in x with degree less than 3. Find a basis for the subspace of $\mathbb{P}_3$ defined as below:

$$\rightarrow S = \{f(x) \in \mathbb{P}_3 : f(x^2) \overset{?}{=} xf(x)\} \leqslant \mathbb{P}_3$$

$f(x) = ax^2 + bx + c.$ $b \in \mathbb{R}$

$f(x^2) = a(x^2)^2 + b(x^2) + c$

$xf(x) = x(ax^2 + bx + c)$

$ax^4 + \cancel{bx^2} + c = ax^3 + \cancel{bx^2} + cx$

$S = \{ f(x) = \cancel{ax^2} + bx + \cancel{c} \ : \ a=0, \ c=0, \ b \in \mathbb{R} \}$ $\Rightarrow ax^4 - ax^3 - cx + c = 0$

$a = 0$ $c = 0$

$= \{ bx \ : \ b \in \mathbb{R} \}$

a typical vector in S.

$b(x) \rightarrow \{x\} \rightarrow$ a basis for S.

Ex/

$S = \{ f(x) \in \mathbb{P}_3 \ : \ f'(3) = f(1) \} \leqslant \mathbb{P}_3$ Find a basis for S

$f(x) = ax^2 + bx + c \longrightarrow f'(x) = 2ax + b$ $f'(3) = 6a + b$

$f(1) = a + b + c$

$6a + \cancel{b} = a + \cancel{b} + c$

$5a - c = 0$ $\rightarrow b \in \mathbb{R}$

$5a = c$

$S = \{ f(x) = ax^2 + bx + c \ : \ 5a - c = 0 \}$

$= \{ f(x) = ax^2 + bx + c \ : \ b \in \mathbb{R}, \ 5a = c \}$

$= \{ ax^2 + bx + 5a \ : \ a, b \in \mathbb{R} \}$

$ax^2 + bx + 5a = a(\underbrace{x^2+5}_{v_1}) + b(\underbrace{x}_{v_2})$

A basis for $S = \{ x, x^2+5 \}$

3. Check the given set of vectors for the two properties separately:
Is the set linearly independent?/ Does it span the given vector space?

→ (a) $\left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$ in $\mathbb{R}^2$. lin. indep ✗ spans $\mathbb{R}^2$ ✓

→ (b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$ in $\mathbb{R}^3$.

→ (c) $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \right\}$ in $\mathbb{R}^{2\times 2}$.

→ (d) $\{2, x^2, x, 2x+3\}$ in $\mathbb{P}_3$. lin. indep ✗ span ✓

lin. indep

$c_1v_1 + c_2v_2 + \dots + c_nv_n = 0_V$

should have only the trivial soln: $c_1 = c_2 = \dots = c_n = 0$

inf many soln ✗

to span V → a typical vector of V.

$r_1v_1 + r_2v_2 + \dots + r_nv_n =$

this system should have a solution for $r_1, r_2, \dots, r_n$ for any "typical"

inf. many ✓
unique soln ✓
no solution possible ⇒ ✗

a) $\dim(\mathbb{R}^2) = 2 < 3$-element set can not be linearly independent.

Span ?

→ $r_1v_1 + r_2v_2 + r_3v_3 = \begin{bmatrix} a \\ b \end{bmatrix}$

$\left[\begin{array}{ccc|c} -2 & 1 & 2 & a \\ 1 & 3 & 4 & b \end{array}\right] \to \dots \to \begin{array}{cccc} r_1 & r_2 & r_3 & \\ \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 4 & b \\ 0 & 7 & 10 & 2b+a \end{array}\right]$ → $r_3 = r \in \mathbb{R}$ → free, $r_1, r_2 \dots$ } ⇒ inf. many solns. ✓

⇒ $\{v_1, v_2, v_3\}$ spans $\mathbb{R}^2$

b) $\dim(\mathbb{R}^3) = 3 < 4$-element set can not be lin. independent.

span?

$r_1v_1 + r_2v_2 + r_3v_3 + rv_4 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$r_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + r_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + r_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + r_4 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 0 & 0 & 0 & b \\ 0 & 1 & 1 & 3 & c \end{array}\right]$ → if $b \neq 0$ this system doesnot have a solution. } ⇒ $\{v_1, v_2, v_3, v_4\}$ is not a spanning set for $\mathbb{R}^3$.

c) $\left\{ \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{v_2}, \underbrace{\cancel{\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}}}_{v_3} \right\}$

lin indep ?
spans $\mathbb{R}^{2\times 2}$?

$\dim(\mathbb{R}^{2\times 2}) = 4 > 3$-element set can not span $\mathbb{R}^{2\times 2}$.

if you can not see this → $2v_1 + 3v_2 = v_3$ $\{v_1, v_2, v_3\}$ is NOT a lin. indep. set.

lin. indep: $c_1v_1 + c_2v_2 + c_3v_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ → $c_1 + 2c_3 = 0$
$c_2 + 3c_3 = 0$

it yo...

lin. indep.:

$c_1 v_1 + c_2 v_2 + c_3 v_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow$

$c_1 + 2c_3 = 0$
$c_2 + 3c_3 = 0$
$c_1 + 2c_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \rightarrow \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$c_3 \rightarrow$ free $\rightarrow$ inf-many solns X not linearly indep..

2. Let $E = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and $F = \left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

(a) Show that E and F are two bases for $\mathbb{R}^3$ $\rightarrow \dim(\mathbb{R}^3) = 3$

$\det(E) \neq 0 \quad \det(F) \neq 0 \quad \Rightarrow$ E, F are 2 bases for $\mathbb{R}^3$

(b) Find the transition matrix from E to F. $= F^{-1}E$

(c) Find the coordinate vector of $v = \begin{bmatrix} -7 \\ 2 \\ -5 \end{bmatrix}$ with respect to E, $[v]_E$. $\longrightarrow [v]_E = E^{-1} v$

(d) Use the transition matrix to find $[v]_F$ $\rightarrow [v]_F = F^{-1}E \; [v]_E$

Find the eigenvalues (λ) and the corresponding eigenspaces (v) for each of the following matrices: $\rightarrow$ eigenvectors and

(b) $\begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix} = A$

$\det(A - \lambda I) = 0 \quad \Rightarrow \quad (A - \lambda I)\vec{x} = 0$

$$\begin{vmatrix} 6-\lambda & -4 \\ 3 & -1-\lambda \end{vmatrix} = (6-\lambda)(-1-\lambda) - (-12) = 0$$

$\lambda^2 - 6\lambda + \lambda - 6 + 12 = 0$

$\lambda^2 - 5\lambda + 6 = 0 \quad (\lambda-2)(\lambda-3) = 0$

$-2 \;\; -3$

$\lambda_1 = 2 \quad \lambda_2 = 3$ are the eigenvalues for A.

for $\lambda_1 = 2$: $(A - \lambda I)\vec{x} = 0$

$$\left[\begin{array}{cc|c} 6-2 & -4 & 0 \\ 3 & -1-2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 3 & -3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = r \in \mathbb{R}$
$\Rightarrow x_1 = r$

solution space $= \{ (r, r) : r \in \mathbb{R} \} \rightarrow$ eigenspace of $\lambda = 2$

$\begin{bmatrix} r \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \underbrace{\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}}$ a basis $\rightarrow$ eigenvector corresponding to $\lambda = 2$.

for $\lambda_2 = 3$:

$$\left[\begin{array}{cc|c} 6-3 & -4 & 0 \\ 3 & -1-3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 3 & -4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow$$

$x_2 = r \in \mathbb{R}$
$x_1 = 4r/3$

solution space $= \{ (4r/3, r) : r \in \mathbb{R} \} \rightarrow$ eigenspace of $\lambda = 3$.

$r \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} \rightarrow$ a basis $\left\{ \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} \right\} \rightarrow$ eigenvector for $\lambda = 3$.

(j) $\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} = A$ $\qquad \det(A - \lambda I) = 0$

$$\begin{vmatrix} -2-\lambda & 0 & 1 \\ 1 & 0-\lambda & -1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = (-2-\lambda)\left[\lambda(1+\lambda)+1\right] - 0 + 1 \cdot \left[1-0\right] = 0$$

$$(-2-\lambda)\left[\lambda^2+\lambda+1\right] + 1 = 0$$

$$-\lambda^3 - 2\lambda^2 - 2\lambda - \lambda^2 - 2 - \lambda + 1 = 0$$

$$-\lambda^3 - 3\lambda^2 - 3\lambda - 1 = 0$$

$$-(\lambda+1)^3 = 0 \qquad \lambda_1 = -1$$

is the only eigenvalue of A.

$\lambda = -1$: $\quad (A - \lambda I)\vec{x} = 0$

(j) $\begin{bmatrix} -2-(-1) & 0 & 1 \\ 1 & 0-(-1) & -1 \\ 0 & 1 & -1-(-1) \end{bmatrix}$ $\qquad \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

$$\rightarrow \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_3 = r \in \mathbb{R}$

$x_2 = 0$

$x_1 + x_2 - x_3 = 0 \Rightarrow x_1 = r$

solution space $= \{ (r, 0, r) : r \in \mathbb{R} \} \rightarrow$ eigenspace for $\lambda = -1$

$\begin{bmatrix} r \\ 0 \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \rightarrow$ eigenvector for $\lambda = -1$

$$A = \begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 2-\lambda & 0 & 0 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix} \to \text{diagonal}$$

$$\det = (2-\lambda)(2-\lambda)(3-\lambda)(4-\lambda) = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 3, \quad \lambda_3 = 4$$

For $\lambda = 2$:

$(A-\lambda I)\vec{x} = 0$

$$\left[\begin{array}{cccc|c} 2-2 & 0 & 0 & 0 & 0 \\ 0 & 2-2 & 0 & 0 & 0 \\ 0 & 0 & 3-2 & 0 & 0 \\ 0 & 0 & 0 & 4-2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = r \in \mathbb{R} \\ x_2 = s \in \mathbb{R} \\ x_3 = 0 \\ x_4 = 0 \end{array}$$

solution space $= \{ (r, s, 0, 0) : r, s \in \mathbb{R} \} \to$ eigenspace for $\lambda = 2$.

$$\begin{bmatrix} r \\ s \\ 0 \\ 0 \end{bmatrix} = r \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ are eigenvectors for $\lambda = 2$.

for $\lambda = 3$:

$$\left[\begin{array}{cccc|c} 2-3 & 0 & 0 & 0 & 0 \\ 0 & 2-3 & 0 & 0 & 0 \\ 0 & 0 & 3-3 & 0 & 0 \\ 0 & 0 & 0 & 4-3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = r \in \mathbb{R} \\ x_4 = 0 \end{array}$$

solution space $= \{ (0, 0, r, 0) : r \in \mathbb{R} \}$

$\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \to$ eigenvector for $\lambda = 3$.

for $\lambda = 4$:

$$\left[\begin{array}{cccc|c} 2-4 & 0 & 0 & 0 & 0 \\ 0 & 2-4 & 0 & 0 & 0 \\ 0 & 0 & 3-4 & 0 & 0 \\ 0 & 0 & 0 & 4-4 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = r \in \mathbb{R} \end{array}$$

solution space $= \{ (0, 0, 0, r) : r \in \mathbb{R} \}$

$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \to$ eigenvector for $\lambda = 4$.